Assignment 1

Due: Monday, Nov 7, 12:15
Hand in at **beginning** of tutorial!

The solutions to this assignment will be discussed in the tutorial on Monday, Nov 7. Therefore, if you want your assignment to be marked, it **has to be handed in at the beginning of the tutorial**. If you can not make it to the tutorial yourself, please arrange for one of your classmates to hand in your assignment.
A full score on this assignment are 50 marks. You can gain up to 10 bonus marks by answering the last question. To gain full marks, explain your answers carefully and prove your claims.

Let $\mathcal{H} \subseteq \{0,1\}^{\mathcal{X}}$ be some hypothesis class over some domain $\mathcal{X}$.

1. **Empirical Risk Minimization**
   Recall that a learner $\mathcal{A}$ is an Empirical Risk Minimizer (ERM) for $\mathcal{H}$ if for all samples $S \in \bigcup_{i=1}^{\infty}(\mathcal{X} \times \{0,1\})^i$, it outputs a function from $\mathcal{H}$ of minimal empirical risk:
   $$\mathcal{A}(S) \in \mathrm{argmin}_{h \in H} L_n(h).$$
   Describe two different learners that are ERM for the class $\mathcal{H}_{\mathrm{rec}}$ of axis aligned rectangles over $\mathcal{X} = \mathbb{R}^d$, defined as
   $$\mathcal{H}_{\mathrm{rec}} := \{h_b \;:\; \boldsymbol{b} \in \mathbb{R}^{2d}\}$$
   where $h_b(\boldsymbol{x}) = 1$ if and only if $x_i \in [b_i, b_{d+i}]$.

   (5 + 5 marks)

2. **Learnability**
   Show that the class of singletons $\mathcal{H}_{\mathrm{sing}}$ is learnable in the realizable case. $\mathcal{H}_{\mathrm{sing}}$ is defined as
   $$\mathcal{H}_{\mathrm{sing}} = \{h \in \{0,1\}^{\mathcal{X}} \;:\; |\{x \in \mathcal{X} \;:\; h(x) = 1\}| \leq 1\}$$

   (10 marks)

3. **VC-dimension**

(a) Let

$$\mathcal{H}_{\leq \mathrm{k}} = \{h \in \{0,1\}^{\mathcal{X}} \ : \ |\{x \in \mathcal{X} \ : \ h(x) = 1\}| \leq k\}.$$

What is the VC-dimension of $\mathcal{H}_{\leq \mathrm{k}}$? How does it depend on the cardinality of the domain $\mathcal{X}$? Prove your claims.

(10 marks)

(b) Let

$$\mathcal{H}_{\mathrm{k}} = \{h \in \{0,1\}^{\mathcal{X}} \ : \ |\{x \in \mathcal{X} \ : \ h(x) = 1\}| = k\}.$$

What is the VC-dimension of $\mathcal{H}_{\mathrm{k}}$? How does it depend on the cardinality of the domain $\mathcal{X}$? Prove your claims.

(10 marks)

(c) Show that adding one function to a hypothesis class can increase the VC-dimension by at most one. That is, for any hypothesis class $\mathcal{H} \subseteq \{0,1\}^{\mathcal{X}}$ and any $h \in \{0,1\}^{\mathcal{X}}$, we have

$$\mathrm{VC}(H \cup \{h\}) \leq \mathrm{VC}(H) + 1.$$

(10 marks)

(d) Show that the inequality in the above question can be tight. That is, give an example of a class $\mathcal{H}$ over some domain and a function h with

$$\mathrm{VC}(H \cup \{h\}) = \mathrm{VC}(H) + 1.$$

(10 bonus marks)