STABILITY ANALYSIS AND CONTROL DESIGN
WITH IMPACTS AND FRICTION

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CONTACT AND CONTROL

- Complex contact scenarios make typical control policies very brittle
- Require fast and accurate sensing of contact state
- Struggle when unexpected contact occurs
- Carefully tuned, ad-hoc approaches near contact events
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Algorithms for finding and verifying controllers robust to contact dynamics?
WHAT MAKES CONTACT HARD?
What makes contact hard?

It's where our intuitive designs struggle
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- Discontinuities
  - $x(t)$
  - $f(x, u)$
  - Continuity of solutions w.r.t initial conditions

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- Uniqueness
- Zeno Phenomena

It's where our intuitive designs struggle
**Models: Dynamics, Impacts, and Friction**

- Joint coordinates \( q \) and velocities \( v \)
  \[
  x = \begin{bmatrix} q \\ v \end{bmatrix}
  \]

- Rigid-body dynamics:
  \[
  H(q)\ddot{q} + C(q, \dot{q}) = Bu + J(q)^T \lambda
  \]

- Contact gap function \( \phi(q) \) and Jacobian \( J_N(q) \)
- Impulsive, inelastic impacts
  \[
  J_N(q)v^+ = 0
  \]

- "Incremental" frictional impacts [Routh 1891]
- Coulomb friction
  \[
  \lambda \in FC
  \]
VERIFICATION AND ANALYSIS

- Lyapunov functions powerful for smooth systems
  - Hybrid versions analyze every contact mode and transition
    Exponential in number of contacts [Papachristodoulou 2009]
- Instead, use Measure Differential Inclusion (MDI) framework [Moreau 1988, Stewart 2000, and others]
  - Exploit complementarity struction
  - Pose tractable SOS programs
LYAPUNOV FUNCTIONS

Capture global or regional stability properties of nonlinear systems

\[ V : \mathbb{R}^n \to \mathbb{R}^+ \]
\[ V(x) \geq 0 \]
\[ \dot{V}(x) = \frac{\partial V}{\partial x} f(x, u) \leq 0 \]

Example: reverse-time Van der Pol oscillator

\[ \dot{x}_1 = -x_2 \]
\[ \dot{x}_2 = x_1 + (x_1^2 - 1)x_2 \]

[Tan and Packard]
Sums-of-squares and Lyapunov Functions

- Fundamentally questions of non-negativity
  - If $p(x)$ a polynomial, $p(x) \geq 0$ is NP-hard
- Replace with a sufficient condition
  $$p(x) = \sum_i a_i^2(x)$$

- Finding $a_i(x)$ is a convex constraint in a Semidefinite Program (SDP)* [Parillo 2000, Lasserre 2001]

  $$\begin{align*}
  \text{find} & \quad Q \succeq 0 \\
  \text{s.t.} & \quad p(x) = m(x)^T Q m(x)
  \end{align*}$$

For some basis $m(x)$ (e.g. monomials up to degree $d$)

*Generalization of linear programming
**S-PROCEDURE**

- Regional analysis for semialgebraic set $\mathcal{F}$
- For example, $\mathcal{F} = \{x : V(x) < \rho\}$

  $$x \in \mathcal{F} \Rightarrow p(x) \geq 0$$

- Sufficient condition:

  $$p(x) - \sigma(x)(V(x) - \rho) \geq 0$$

  $$\sigma(x) \geq 0$$

$\rho$-sublevel set of $V$ in blue
PASSIVE RIMLESS WHEEL

- 5 state model
- 2 contact points
- Exhibits Zeno phenomena

Goal: verify stable region

[Posa, Tobenkin, and Tedrake (TAC 2016)], [Posa, Tobenkin, and Tedrake (HSCC 2013)]
LYAPUNOV FUNCTIONS FOR MDIS

- Measure differential inclusion modeling framework, instead of ODE
- Encompasses discontinuities and non-uniqueness

\[ dq = v(t)dt \]
\[ dv = \dot{v}(t)dt + (v^+(t) - v^-(t))d\eta(t) \]

- \( q(t) \) continuous and \( v(t) \) of locally bounded variation
- Dynamics from set-valued functions
  - Simulation requires selection from set
  - For verification, take permissive view

Prove
\[ dV = \dot{V}(t) + (V^+(t) - V^-(t)) \leq 0 \]
for all possible scenarios
HOW TO EFFICIENTLY VERIFY $dV \leq 0$?

$(q, v, \lambda)$ admissible $\Rightarrow dV(q, v, \lambda) \leq 0$
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$(q, v, \lambda)$ admissible $\Rightarrow dV(q, v, \lambda) \leq 0$

$$
\dot{V} = \frac{\partial V}{\partial q} v + \frac{\partial V}{\partial v} H^{-1}(C + Bu)
$$

No contact

$$
\dot{V} = \frac{\partial V}{\partial q} v + \frac{\partial V}{\partial v} H^{-1}(C + Bu + J^T \lambda)
$$

Continuous contact

$$
V^+ - V^- = \int_0^\Lambda \frac{\partial V}{\partial v} H^{-1} J^T \lambda d\lambda
$$

Routh impact
HOW TO EFFICIENTLY VERIFY $dV \leq 0$?

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$$\dot{V} = \frac{\partial V}{\partial q} v + \frac{\partial V}{\partial v} H^{-1} (C + Bu + J^T \lambda)$$  Continuous contact

$$V^+ - V^- = \int_{0}^{\Lambda} \frac{\partial V}{\partial v} H^{-1} J^T \lambda d\lambda$$  Routh impact

Observation: sufficient to check

$$\frac{\partial V}{\partial q} v + \frac{\partial V}{\partial v} H^{-1} (C + Bu) \leq 0$$

$$\frac{\partial V}{\partial v} H^{-1} J^T \lambda \leq 0$$
LEVERAGING STRUCTURE

What \((q, v, \lambda)\) are admissible?
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What \((q, v, \lambda)\) are admissible?

- Leverage complementarity formulation
  - Non-penetration: \(\phi(q) \geq 0\)
  - Friction cone: \(\lambda \in \mathcal{FC}\)
  - Contact mode logic: \(\phi(q) > 0 \Rightarrow \lambda = 0\)
- Semialgebraic in \((q, v, \lambda)\)

\[
\begin{align*}
\phi(q) & \geq 0 \\
\lambda_N & \geq 0 \\
\phi(q)\lambda_N & = 0
\end{align*}
\]
LEVERAGING STRUCTURE

What \((q, v, \lambda)\) are admissible?

\[
\begin{align*}
\phi, \lambda_N & \geq 0 \quad \text{Non-penetration, normal force} \\
(J_N v) \lambda_N & \leq 0 \quad \text{Normal dissipation} \\
(J_f v) \lambda_f & \leq 0 \quad \text{Frictional dissipation} \\
\phi \lambda_N & = 0 \quad \text{No force at distance} \\
\mu^2 \lambda_N^2 & - \lambda_f^2 \geq 0 \quad \text{Friction cone} \\
(\mu^2 \lambda_N^2 - \lambda_f^2)(J_f v) & = 0 \quad \text{Sliding friction}
\end{align*}
\]
**TRACTABLE OPTIMIZATION PROGRAM**

Theorem [TAC 2016]: For $m$ contacts, can **eliminate** normal force variables and verify contacts **independently**

- Proof exploits continuity and homogeneity in $\lambda$
- Reduced conditions:

$$\frac{\partial V}{\partial q} \nu + \frac{\partial V}{\partial v} H^{-1}(C + Bu) \leq 0 \iff \phi \geq 0$$

$$\frac{\partial V}{\partial v} H^{-1}(J_{N,i}^T + J_{f,i}^T \lambda_{f,i}) \leq 0 \iff \left\{ \begin{array}{l}
\phi \geq 0 \\
\phi_i = 0 \\
J_{N,i} \nu \leq 0 \\
(J_f \nu) \lambda_{f,i} \leq 0 \\
\mu^2 - \lambda_{f,i}^2 \leq 0 \\
(\mu^2 - \lambda_{f,i}^2) J_{f,i} \nu = 0 \end{array} \right. \quad i=1,..,n$$
# Tractable Optimization Program

For $n$-dimensional state, $m$ contacts, and polynomial degree $d$,

<table>
<thead>
<tr>
<th>Hybrid*</th>
<th>multiple Lyapunov functions</th>
<th>$\mathcal{O}(3^m)$</th>
<th>$\mathcal{O}(3^m n^d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MDI</td>
<td>single Lyapunov function</td>
<td>$\mathcal{O}(m^2)$</td>
<td>$\mathcal{O}(m^2 (n + 1)^d)$</td>
</tr>
</tbody>
</table>

*Exact count depends on formulation of sliding and sticking contact
CONTROL DESIGN AND VERIFICATION

- 7 state model
- 2 contact points
- 1 input
- Control design bilinear in $V(x)$ and $u(x)$
- Solve via alternating sequence of SOS programs

What is possible with a continuous feedback policy $u(x)$?

Goal: maximize stable region
**CONTROL DESIGN AND VERIFICATION**

- Sampled 2D slice of state space and simulate
- Verified region in blue
- Stable samples as red dots
Unsafe Region Avoidance
**Discussion**

- A step toward control and verification in complex contact scenarios
- Restricted to fairly simple models
  - Beyond capability of hybrid formulation
  - SOS/SDP relatively immature
  - Introduce model/terrain uncertainty at computational cost
- Preliminary work in exploiting simple models for push recovery
- Relevant publications